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On the existence of solutions to the Benjamin-Ono equation

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1. Intrduction

In this talk, we consider the existence and the uniqueness of solutions to the Benjamin-Ono(BO) equation,

$$(1)\qquad \begin{cases} \partial_t u + \mathrm{H}\partial_x^2 u + \frac{1}{2}\partial_x(u^2) = 0, & \text{in } \mathbb{R}\times\mathbb{R}, \\ u(0,x) = \phi(x), & \text{in } \mathbb{R}, \end{cases}$$

where H is the Hilbert transform which is defined by

$$\mathrm{H}f = \mathrm{p.v.}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(y)}{x-y}dy = \mathcal{F}^{-1}(-i\, sgn(\xi))\mathcal{F}f,$$

$\mathcal{F}$ denotes the Fourier transform with respect to x and $sgn(\xi)$ denotes the signature of ξ. BO equation describes long internal waves in deep stratified fluids [3], [11]. As well as the Korteweg-de Vries equation, BO equation is completely integrable [1]. Hence if the initial function is real valued, this equation has infinitely many conservative quantities. The Cauchy problem of this equation is extensively studied by using this property [2], [5], [6], [9], [12], [13] and references therein. It is known that this equation is locally well-posed for real valued initial function in Sobolev space $H^s(\mathbb{R})$ for $s \geq 1$ and globally well-posed for $s = 1$ and $s \geq 3/2$.

On the other hand, Molinet-Saut-Tzvetkov [10] has shown that for any $s \in \mathbb{R}$ the Benjamin-Ono equation cannot be solved by the iteration method in H^s.

The aim of this note is to show the existence, the uniqueness and the continuous dependency of the initial data of solutions to the Benjamin-Ono equation by the iteration method for some Sobolev spaces mixed between homogenious and inhomogenious Sobolev spaces which is defined in the definition 2. In this direction, N. Kita and J. Segata[8] has recently shown the wellposedness of solutions for the weighted Sobolev space by the iteration method, which consists of functions satisfying that $\phi \in H^s$ with $s > 1$ and $\langle x\rangle^\alpha\phi \in H^{s_1}$ with $s_1 + \alpha < s$, $1/2 < s_1$ and $1/2 < \alpha < 1$.

In our result, we assume the smallness of the initial function, but the result of this paper may be a first step to show local well-posedness of BO equation for the usual Sobolev space for $s > 1/2$. Our approach is to use so called Fourier restriction norm which is developped by [4] and [7]. Our function space with Fourier restriction norm is the following.

Definition 1. Let s_1, s_2, b_1 and b_2 be real numbers. We define a function space $X^{s_1,s_2}_{b_1,b_2}$ as follows;

$$(2)\quad X^{s_1,s_2}_{b_1,b_2} = \{f \in \mathcal{S}'(\mathbb{R}^2);\ \|f\|_{X^{s_1,s_2}_{b_1,b_2}} = \|\langle\xi\rangle^{s_1}|\xi|^{s_2}\langle\tau+\xi^2\rangle^{b_1}\langle\tau-\xi^2\rangle^{b_2}\hat{f}(\tau,\xi)\|_{L^2_{\tau,\xi}} < +\infty\Big\}.$$

Here $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$ and $\hat{f}(\tau,\xi)$ is the Fourier transform of $f(t,x)$ with respect to space and time variables.

We shall find a solution to the associate integral equation of

$$(3)\qquad u(t) = U(t)\phi + \int_0^t U(t-s)\partial_x(u(s)^2)ds,$$

instead of the intial value problem (1) directly. Here $U(t)\phi = e^{(-t\mathrm{H}\partial_x^2)}\phi = \mathcal{F}^{-1}e^{(-it\xi|\xi|)}\mathcal{F}\phi$. Let ψ be a function in $C_0^\infty(\mathbb{R})$ with $0 \le \psi \le 1$, $\psi(t) = 1$ for $|t| \le 1$ and $\psi(t) = 0$ for $|t| \ge 2$. We consider the following integral equation,

$$(4)\qquad u(t,x) = \psi(t)U(t)\phi + \psi(t)\int_0^t U(t-s)\partial_x(u(s)^2)ds.$$

Definition 2. Let s_1 and s_2 be real numbers. Function space $H^{s_1,s_2}(\mathbb{R})$ is defined by

(5)

$$H^{s_1,s_2}(\mathbb{R}) = \{g(x) \in \mathcal{S}'(\mathbb{R}); \|g\|_{H^{s_1,s_2}} = \|\langle\xi\rangle^{s_1}|\xi|^{s_2}\hat{g}(\xi)\|_{L^2} < +\infty\}.$$

We write $H^{s_1,s_2} = H^{s_1,s_2}(\mathbb{R})$ for abbreviation. Our main theorem is the following.

Theorem 1. *Suppose that $\delta > 0$, $\phi \in H^{1+\delta,-1/2}(\mathbb{R})$ and $\|\phi\|_{H^{1+\delta,-1/2}}$ is sufficiently small. Then there exists a unique solution $u(t,x)$ to the integral equation (4) in $X^{\delta,-1/2}_{1/2,1/2}$. Moreover, we have*

$$(6)\qquad \|u_1(t,x)-u_2(t,x)\|_{X^{\delta,-1/2}_{1/2,1/2}} \le C\|\phi_1-\phi_2\|_{H^{1+\delta,-1/2}},$$

where u_j is a solution to the equation (4) with initial data ϕ_j for $j = 1, 2$.

Remark 1. *Since $\langle\xi\rangle^{2b}|\xi|^{-1/2} \approx |\xi|^{2b-1/2}$ for $|\xi|$ large, functions in $H^{2b,-1/2}$ have the same regularity as functions in $H^{2b-1/2}$.*

Remark 2. *The space $X^{0,-1/2}_{b,b}$ is included by the space $C(\mathbb{R}; H^{2b,-1/2})$, which is shown in Lemma 6.*

Remark 3. *In [10], it is pointed out that the interaction between high energy and low energy disturbs the Picard's iteration method for the BO equation in usual Sobolev space. In our result, we avoid this difficulty to use the space $H^{2b,-1/2}$. Low energy part of functions in $H^{2b,-1/2}$ is small since the Fourier transform of functions in $H^{2b,-1/2}$ may vanish at 0.*

Through the paper, $I \lesssim J$ denotes that there exists a harmless constant $C > 0$ such that $I \leq CJ$. $I \sim J$ denotes that there exist harmless constants $C_1, C_2 > 0$ such that $C_1 J \leq I \leq C_2 J$. For abbreviation, we write $\{h(\tau,\xi) \leq 0\}$ as $\{(\tau,\xi) | h(\tau,\xi) \leq 0\}$.

2. Preliminaries

In this section, we prepare several lemmas for the proof of the main theorem. The following lemma is used in [7].

Lemma 1. *If $\alpha > 1$ and $a, b \in \mathbb{R}$, then*

$$\int_{-\infty}^{\infty} \frac{1}{\langle \xi - a\rangle^{\alpha} \langle \xi - b\rangle^{\alpha}} d\xi \leq C\langle a - b\rangle^{-\alpha}.$$

Lemma 2. *If $a, b \in \mathbb{R}$, then for all $\epsilon > 0$ there exists a constant $C > 0$ such that*

$$\int_{-\infty}^{\infty} \frac{1}{\langle \xi - a\rangle \langle \xi - b\rangle} \leq C_{\epsilon}\langle a - b\rangle^{-1+\epsilon}.$$

The proofs of these lemmas can be done elementarily. So we omit the proofs.

Lemma 3. *If $\alpha \geq 1/2$, $\beta \geq 0$ with $\alpha + \beta/2 > 1$ and b, $c \in \mathbb{R}$, then we have*

$$\int_{-\infty}^{\infty} \frac{1}{\langle \xi^2 + b\xi + c\rangle^{\alpha} \langle \xi\rangle^{\beta}} d\xi \lesssim \langle c - b^2/4\rangle^{-1/2}.$$

Proof. Changing variable as $\xi' = \xi^2 + b\xi + c$ in the right hand side, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{\langle \xi^2 + b\xi + c\rangle^{\alpha} \langle \xi\rangle^{\beta}} d\xi = \int_{-\infty}^{-b/2} \cdots + \int_{-b/2}^{\infty} \cdots \\
=&\frac{1}{2}\int_{c-b^2/4}^{\infty} \frac{d\xi'}{\langle \xi'\rangle^{\alpha} \langle b/2 + \sqrt{\xi' - (c - b^2/4)}\rangle^{\beta} |\xi' - (c - b^2/4)|^{1/2}} \\
&\quad + \frac{1}{2}\int_{c-b^2/4}^{\infty} \frac{d\xi'}{\langle \xi'\rangle^{\alpha} \langle \sqrt{\xi' - (c - b^2/4)} - b/2\rangle^{\beta} |\xi' - (c - b^2/4)|^{1/2}} \\
=&\frac{1}{2} I_1 + \frac{1}{2} I_2.
\end{aligned}$$

We can assume without loss of generality that $|c - b^2/4| \geq 1$. If $c - b^2/4 \geq 1$, then

$$\begin{aligned}
I_1 &\lesssim \langle c - b^2/4\rangle^{-1/2} \\
&\times \int_{c-b^2/4}^{\infty} \frac{d\xi'}{\langle \xi'\rangle^{\alpha-1/2} \langle b/2 + \sqrt{\xi' - (c - b^2/4)}\rangle^{\beta} |\xi' - (c - b^2/4)|^{1/2}} \\
&\lesssim \langle c - b^2/4\rangle^{-1/2}.
\end{aligned}$$

If $c - b^2/4 \leq -1$, then

$$
\begin{aligned}
I_1 &= \int_{c-b^2/4}^{(c-b^2/4)/2} \cdots + \int_{(c-b^2/4)/2}^{\infty} \cdots \\
&\lesssim \langle c - b^2/4 \rangle^{-1/2} \\
&\times \left\{ \int_{c-b^2/4}^{(c-b^2/4)/2} \frac{d\xi'}{\langle \xi' \rangle^{\alpha-1/2} \langle b/2 + \sqrt{\xi' - (c-b^2/4)} \rangle^{\beta} |\xi' - (c-b^2/4)|^{1/2}} \right. \\
&\qquad \left. + \int_{(c-b^2/4)/2}^{\infty} \frac{d\xi'}{\langle \xi' \rangle^{\alpha} \langle b/2 + \sqrt{\xi' - (c-b^2/4)} \rangle^{\beta}} \right\} \\
&\lesssim \langle c - b^2/4 \rangle^{-1/2}.
\end{aligned}
$$

The similar argument as above is valid for I_2. □

3. Linear estimates

In this section, we prepare some estimates of the evolution operator $U(t) = exp(tH\partial_x^2)$ for the linear part of the Benjamin-Ono equation.

Lemma 4. *For $\phi \in \mathcal{S}(\mathbb{R})$, we have*

$$
\|\psi(t)U(t)\phi\|_{X_{b,b}^{s_1,s_2}} \leq C\|\phi\|_{H^{s_1+2b,s_2}},
$$

Where $\|\phi\|_{H^{2b,-\rho}} = \||\xi|^{-\rho}\langle \xi \rangle^{2b}\hat{\phi}(\xi)\|_{L^2}$.

Proof.

$$
\begin{aligned}
&\|\psi(t)U(t)\phi\|_{X_{b,b}^{s_1,s_2}} \\
=&\|\langle \xi \rangle^{s_1}|\xi|^{s_2}\langle \tau + \xi^2 \rangle^{b}\langle \tau - \xi^2 \rangle^{b} \int \psi(t)e^{-it\xi|\xi|-it\tau}dt\hat{\phi}(\xi)\|_{L^2} \\
=&\|\langle \xi \rangle^{s_1}|\xi|^{s_2}\langle \tau + \xi^2 \rangle^{b}\langle \tau - \xi^2 \rangle^{b}\hat{\psi}(\tau + \xi|\xi|)\hat{\phi}(\xi)\|_{L^2} \\
\lesssim&\|\chi_{\{\xi \geq 0\}}\langle \xi \rangle^{s_1+2b}|\xi|^{s_2}\langle \tau + \xi^2 \rangle^{2b}\psi(\tau + \xi^2)\hat{\phi}(\xi)\|_{L^2} \\
&+ \|\chi_{\{\xi < 0\}}\langle \xi \rangle^{s_1+2b}\langle \xi \rangle^{s_2}\langle \tau - \xi^2 \rangle^{2b}\psi(\tau - \xi^2)\hat{\phi}(\xi)\|_{L^2} \\
\lesssim&\|\langle \tau \rangle^{2b}\hat{\psi}(\tau)\|_{L^2}\|\langle \xi \rangle^{s_1+2b}|\xi|^{s_2}\hat{\phi}(\xi)\|_{L^2} \\
\lesssim&\|\phi\|_{H^{s_1+2b,s_2}}.
\end{aligned}
$$

□

Lemma 5. *For $f(t,x) \in \mathcal{S}(\mathbb{R}^2)$, we have*

$$
\|\psi(t)\int_0^t U(t-s)f(s,x)ds\|_{X_{1/2,1/2}^{s_1,s_2}} \leq C\left(\|f\|_{X_{-1/2,1/2}^{s_1,s_2}} + \|f\|_{X_{1/2,-1/2}^{s_1,s_2}} + \|f\|_{Y^{s_1+1,s_2}}\right), \tag{7}
$$

where

$$(8)\quad \|f\|_{Y^{s_1+1,s_2}} = \left(\int_{-\infty}^{\infty} \langle \xi \rangle^{2s_1+2} |\xi|^{2s_1} \left(\int_{-\infty}^{\infty} \frac{|\hat{f}(\tau,\xi)|}{\langle \tau + \xi|\xi| \rangle} d\tau \right)^2 d\xi \right)^{1/2}.$$

The proof of Lemma 5 can be done by the same manner as in Kenig-Ponce-Vega [7].

Lemma 6. *For* $0 < \forall \delta' < \delta$, *we have*

$$(9)\qquad X^{\delta,-1/2}_{1/2,1/2} \subset C(\mathbb{R}; H^{1+\delta',-1/2}).$$

Proof. It suffices to show that there exists a positive constant C such that

$$(10)\qquad \sup_t \|u(t,\cdot)\|_{H^{1+\delta',-1/2}} \le C\|u\|_{X^{\delta,-1/2}_{1/2,1/2}}$$

for $u \in \mathcal{S}$. We denote the Fourier transform of u with respect to x by $\tilde{u}(t,\xi)$. Since $\tilde{u}(t,\xi) = 1/\sqrt{2\pi}\int \hat{u}(\tau,\xi)e^{it\tau}d\tau$, we have

$$(11)\qquad \|u(t,\cdot)\|^2_{H^{1+\delta',-1/2}} = \|\langle\xi\rangle^{1+\delta'}|\xi|^{-1/2}\tilde{u}(t,\xi)\|^2_{L^2}$$

$$(12)\qquad \le \int \langle\xi\rangle^{2+2\delta'}|\xi|^{-1}\left|\int |\hat{u}(\tau,\xi)|d\tau\right|^2 d\xi.$$

Schwarz's inequality shows that

$$\begin{aligned}
&\int |\hat{u}(\tau,\xi)|d\tau \\
&= \int_0^{\infty} \langle\tau-\xi^2\rangle^{-(1+\epsilon)/2}\langle\tau-\xi^2\rangle^{(1+\epsilon)/2}|\hat{u}(\tau,\xi)|d\tau \\
&\qquad + \int_{-\infty}^{0} \langle\tau+\xi^2\rangle^{-(1+\epsilon)/2}\langle\tau+\xi^2\rangle^{(1+\epsilon)/2}|\hat{u}(\tau,\xi)|d\tau \\
&= \left(\int_0^{\infty}\langle\tau-\xi^2\rangle^{-(1+\epsilon)}d\tau\right)^{1/2}\left(\int_0^{\infty}\langle\tau-\xi^2\rangle^{1+\epsilon}|\hat{u}(\tau,\xi)|d\tau\right)^{1/2} \\
&+ \left(\int_{-\infty}^{0}\langle\tau+\xi^2\rangle^{-(1+\epsilon)}d\tau\right)^{1/2}\left(\int_{-\infty}^{0}\langle\tau+\xi^2\rangle^{1+\epsilon}|\hat{u}(\tau,\xi)|d\tau\right)^{1/2} \\
&= \left(\int_0^{\infty}\langle\tau\rangle^{-(1+\epsilon)}d\tau\right)^{1/2}\left\{\left(\int_0^{\infty}\langle\tau-\xi^2\rangle^{1+\epsilon}|\hat{u}(\tau,\xi)|d\tau\right)^{1/2}\right. \\
&\qquad \left. + \left(\int_{-\infty}^{0}\langle\tau+\xi^2\rangle^{1+\epsilon}|\hat{u}(\tau,\xi)|d\tau\right)^{1/2}\right\}.
\end{aligned}$$

Since $\langle\xi\rangle^2, \langle\tau-\xi^2\rangle \le \langle\tau+\xi^2\rangle$ for $\tau \ge 0$ and $\langle\xi\rangle^2, \langle\tau+\xi^2\rangle \le \langle\tau-\xi^2\rangle$ for $\tau \le 0$, we have with $\epsilon = 2(\delta-\delta')$

$$
\begin{aligned}
&\|u(t,\cdot)\|_{H^{1+\delta',-1/2}} \\
&\le C\int_{-\infty}^{\infty}\langle\xi\rangle^{2+2\delta'}|\xi|^{-1}\Big\{\int_0^{\infty}\langle\tau-\xi^2\rangle^{1+\epsilon}|\hat u(\tau,\xi)|d\tau \\
&\qquad + \int_{-\infty}^{0}\langle\tau+\xi^2\rangle^{1+\epsilon}|\hat u(\tau,\xi)|d\tau\Big\}\, d\xi \\
&\le C\int_{-\infty}^{\infty}\langle\xi\rangle^{2\delta'+\epsilon}|\xi|^{-1}\Big\{\int_0^{\infty}\langle\tau+\xi^2\rangle^{1-\epsilon}\langle\tau-\xi^2\rangle^{1+\epsilon}|\hat u(\tau,\xi)|d\tau \\
&\qquad + \int_{-\infty}^{0}\langle\tau-\xi^2\rangle^{1-\epsilon}\langle\tau+\xi^2\rangle^{1+\epsilon}|\hat u(\tau,\xi)|d\tau\Big\} \\
&\le 2C\|u\|^2_{X^{\delta,-1/2}_{1/2,1/2}}.
\end{aligned}
$$

For $t, t' \ge 0$, the same calculation as above yields

$$
\begin{aligned}
&\|u(t,\cdot)-u(t',\cdot)\|^2_{H^{1+\delta',-1/2}} \\
&\le \int \langle\xi\rangle^{2+2\delta'}|\xi|^{-1}\left|\int |e^{i\tau t}-e^{i\tau t'}||\hat u(\tau,\xi)|d\tau\right|^2 d\xi \\
&\le 2C\int\int |e^{i\tau t}-e^{i\tau t'}|\langle\tau+\xi^2\rangle\langle\tau-\xi^2\rangle|\xi|^{-1}\langle\xi\rangle^{1+\delta}|\hat u(\tau,\xi)|^2 d\tau d\xi.
\end{aligned}
$$

Lebesgue's dominated convergent theorem implies that

$$\lim_{t'\to t}\|u(t,\cdot)-u(t',\cdot)\|^2_{H^{1+\delta',-1/2}} = 0.$$

Hence we have (10). □

4. Bilinear estimates

In order to prove the main theorem, we prepare the following two propositions.

Proposition 1. *Let $\delta > 0$. Then there exists a positive constant C such that*

$$\|\partial_x(fg)\|_{X^{\delta,-1/2}_{1/2,-1/2}} \le C\|f\|_{X^{\delta,-1/2}_{1/2,1/2}}\cdot\|g\|_{X^{\delta,-1/2}_{1/2,1/2}}, \tag{13}$$

$$\|\partial_x(fg)\|_{X^{\delta,-1/2}_{-1/2,1/2}} \le C\|f\|_{X^{\delta,-1/2}_{1/2,1/2}}\cdot\|g\|_{X^{\delta,-1/2}_{1/2,1/2}} \tag{14}$$

are valid for $f, g \in \mathcal{S}$. If $f, g \in X^{\delta,-1/2}_{1/2,1/2}$, then $\partial_x(fg)$ is in $X^{\delta,-1/2}_{1/2,-1/2}$ and the inequalities (13) *and* (14) *are valid.*

Proposition 2. *Let $\delta > 0$. For $f, g \in \mathcal{S}$, we have*

$$\|\partial_x(fg)\|_{Y^{1+\delta,-1/2}} \le \|f\|_{X^{\delta,1/2}_{1/2,1/2}}\cdot\|g\|_{X^{\delta,1/2}_{1/2,1/2}},$$

where $\|f\|_{Y^{1+\delta,-1/2}}$ is the quantity defined in (8).

We divide $\mathbb{R}^4$ into several subsets and in each subset D, it suffices to show for Proposition 1 that

$$(15)\quad I(D) = \sup_{\tau,\xi} \frac{|\xi|\langle\tau+\xi^2\rangle}{\langle\xi\rangle^{2\delta}\langle\tau-\xi^2\rangle} \times \iint_{\mathbb{R}^2} \frac{\langle\xi-\xi'\rangle^{-2\delta}|\xi-\xi'|\langle\xi'\rangle^{-2\delta}|\xi'|d\tau'd\xi'}{\langle\tau-\tau'+(\xi-\xi')^2\rangle\langle\tau-\tau'-(\xi-\xi')^2\rangle\langle\tau'+\xi'^2\rangle\langle\tau'-\xi'^2\rangle} < \infty,$$

or

$$(16)\quad J(D) = \sup_{\tau',\xi'} \frac{|\xi'|^{1/2}\langle\xi'\rangle^{2\delta}}{\langle\tau'+\xi'^2\rangle\langle\tau'-\xi'^2\rangle} \times \iint_{\mathbb{R}^2} \frac{\langle\xi-\xi'\rangle^{-2\delta}|\xi-\xi'|\langle\xi\rangle^{-2\delta}|\xi|d\tau d\xi}{\langle\tau-\tau'+(\xi-\xi')^2\rangle\langle\tau-\tau'-(\xi-\xi')^2\rangle\langle\tau+\xi^2\rangle\langle\tau-\xi^2\rangle} < \infty.$$

and for Proposition 2 that

$$(17)\quad \tilde{I}(D) = \sup_{\xi} |\xi|\langle\xi\rangle^{2+2\delta} \int \frac{1}{\langle\tau+\xi|\xi|\rangle^2} \times \iint_{\mathbb{R}^2} \frac{\langle\xi-\xi'\rangle^{-2\delta}|\xi-\xi'|\langle\xi'\rangle^{-2\delta}|\xi'|d\tau'd\xi'd\tau}{\langle\tau-\tau'+(\xi-\xi')^2\rangle\langle\tau-\tau'-(\xi-\xi')^2\rangle\langle\tau'+\xi'^2\rangle\langle\tau'-\xi'^2\rangle} < \infty,$$

or

$$(18)\quad \tilde{J}(D) = \sup_{\tau',\xi'} \frac{|\xi'|\langle\xi'\rangle^{2\delta}}{\langle\tau'+\xi'^2\rangle\langle\tau'-\xi'^2\rangle} \times \iint_{\mathbb{R}^2} \frac{\langle\xi-\xi'\rangle^{-2\delta}|\xi-\xi'|\langle\xi\rangle^{2+2\delta}|\xi|d\tau d\xi}{\langle\tau-\tau'+(\xi-\xi')^2\rangle\langle\tau-\tau'-(\xi-\xi')^2\rangle\langle\tau+\xi|\xi|\rangle^{1-\epsilon}} < \infty$$

for some sufficiently small $\epsilon > 0$. To prove the above propositions, we use the following inequalities:

$$\begin{aligned}
|\xi||\xi'| &\le \frac{3}{2}\max\left(|\tau-\xi^2|,\ |\tau-\tau'-(\xi-\xi')^2|,\ |\tau'+\xi'^2|\right)\\
|\xi'||\xi-\xi'| &\le \frac{3}{2}\max\left(|\tau-\xi^2|,\ |\tau-\tau'-(\xi-\xi')^2|,\ |\tau'-\xi'^2|\right)\\
|\xi||\xi-\xi'| &\le \frac{3}{2}\max\left(|\tau-\xi^2|,\ |\tau-\tau'+(\xi-\xi')^2|,\ |\tau'-\xi'^2|\right)\\
|\tau| &\le 2\max\left(|\tau-\tau'+(\xi-\xi')^2|, |\tau-\tau'-(\xi-\xi')^2|, |\tau'+\xi'^2|, |\tau'-\xi'^2|\right)\\
|\xi'|^2 &\le \max\left(|\tau'+\xi'^2|,\ |\tau'-\xi'^2|\right)\\
|\xi-\xi'|^2 &\le \max\left(|\tau-\tau'+(\xi-\xi')^2|,\ |\tau-\tau'-(\xi-\xi')^2|\right)
\end{aligned}$$

The identity $-2\xi\xi' = \tau - \xi^2 - (\tau - \tau' - (\xi - \xi')^2) - (\tau' + \xi'^2)$ implies the first inequality. Other inequalities are proven by the same way.

The proof of Proposition 1 and Proposition 2 can be done by dividing $\mathbb{R}^4$ with respect to $|\xi|$, $|\xi'|$, $|\xi-\xi'|$, $|\tau-\xi^2|$, $|\tau'-\xi'^2|$, $|\tau-\tau'-(\xi-\xi')^2|$, $|\tau+\xi^2|$, $|\tau'+\xi'^2|$ and $|\tau-\tau'-(\xi-\xi')^2|$.

5. Proof of Theorem 1

In this section, we prove Theorem 1 by combining Lemma 4, Lemma 5 and propositions 1-2 .

Proof of Theorem 1. Let M be a mapping from $X^{\delta,-1/2}_{1/2,1/2}$ to itself defined by

$$Mu = \psi(t)U(t)\phi + \psi(t)\int_0^t U(t-s)\psi(s)\partial_x(u(s)^2)ds. \tag{19}$$

Lemma 4, Lemma 5 and Propositions 1-2 assure that M is well defined on $X^{\delta,-1/2}_{1/2,1/2}$.

First we show that M is a contraction mapping on X_δ with some small $\delta > 0$ if $\|\phi\|_{H^{2b,-1/2}}$ is sufficiently small, where $X_\delta = \{u \in X^{\delta,-1/2}_{1/2,1/2} | \|u\|_{X^{\delta,-1/2}_{1/2,1/2}} \leq \delta\}$. From Lemma 4, Lemma 5 and Propositions 1-2, we have

$$\begin{aligned}
&\|Mu\|_{X^{\delta,-1/2}_{1/2,1/2}}\\
&\leq C_1\|\phi\|_{H^{1+\delta,-1/2}} + C_2\left(\|\psi\partial_x(u^2)\|_{X^{\delta,-1/2}_{-1/2,1/2}} + \|\psi\partial_x(u^2)\|_{X^{0,-1/2}_{b,b-1}}\right)\\
&\leq C_1\|\phi\|_{H^{1+\delta,-1/2}} + C_2C_3\|u\|^2_{X^{\delta,-1/2}_{1/2,1/2}}\\
&\leq C_1\|\phi\|_{H^{1+\delta,-1/2}} + C_2C_3\delta^2.
\end{aligned}$$

If $\|\phi\|_{H^{1+\delta,-1/2}} \leq \delta/(2C_1)$ and $\delta \leq 1/(2C_2C_3)$, then $\|Mu\|_{X^{\delta,-1/2}_{1/2,1/2}} \leq 1/2\delta + 1/2\delta = \delta$. Let $u, v \in X_\delta$. The same calculation as above shows that

$$\begin{aligned}
&\|Mu - Mv\|_{X^{\delta,-1/2}_{1/2,1/2}}\\
&\leq C_2\left(\|\psi\partial_x\{(u+v)(u-v)\}\|_{X^{\delta,-1/2}_{-1/2,1/2}} \|\psi\partial_x\{(u+v)(u-v)\}\|_{X^{\delta,-1/2}_{-1/2,1/2}}\right)\\
&\leq C_2C_3\left(\|u\|_{X^{\delta,-1/2}_{1/2,1/2}} + \|v\|_{X^{\delta,-1/2}_{1/2,1/2}}\right)\|u-v\|_{X^{\delta,-1/2}_{-1/2,1/2}}\\
&\leq 2C_2C_3\delta\|u-v\|_{X^{\delta,-1/2}_{-1/2,1/2}}.
\end{aligned}$$

If we take $\delta \leq 1/(4C_2C_3)$, then $\|u-v\|_{X^{\delta,-1/2}_{1/2,1/2}} \leq 1/2\|u-v\|_{X^{\delta,-1/2}_{1/2,1/2}}$. Thus M is a contraction mapping on X_δ if $\delta < 1/(4C_2C_3)$ and $\|\phi\|_{H^{1+\delta,-1/2}} \leq \delta/(2C_1)$. Hence M has a unique fixed point in X_δ.

Next we show the inequality (6). Let u_1 and u_2 be solutions to (4) in X_δ with initial data ϕ_1 and ϕ_2 respectively. The same calculation as in the above shows

$$\begin{aligned}\|u_1 - u_2\|_{X^{\delta,-1/2}_{1/2,1/2}} &\le C_1\|\phi_1 - \phi_2\|_{H^{1+\delta,-1/2}} + 2C_2C_3\delta\|u - v\|_{X^{\delta,-1/2}_{-1/2,1/2}} \\ &\le C_1\|\phi_1 - \phi_2\|_{H^{1+\delta,-1/2}} + \frac{1}{2}\|u - v\|_{X^{\delta,-1/2}_{-1/2,1/2}}.\end{aligned}$$

This shows $\|Mu_1 - Mu_2\|_{X^{\delta,-1/2}_{1/2,1/2}} \le 2C_1\|\phi_1 - \phi_2\|_{H^{1+\delta,-1/2}}$. □

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